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REVISITS FOR TRANSIENT RANDOM WALK

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Abstract. For a random walk on the integers define R_n as the number of (distinct) states visited in the first n steps and Z_n as the number of states visited in the first n steps which are never revisited. Here we deal with transient walks. The increments of Z_n form a stationary process and various central limit results and an iterated logarithm result are obtained for Z_n from known results on stationary processes. Furthermore, the limit behaviour of R_n is closely related to that of Z_n; this relationship is elucidated and corresponding limit results for R_n are then read off from those for Z_n.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables taking only integer values and write $S_0 = 0$, $S_n = \Sigma_{i=1}^n X_i$, $n \geqslant 1$, so that $\{S_n\}$ generates a random walk. Define R_n as the cardinality of the set $\{S_1, S_2, \ldots, S_n\}$ and

$$V_k = \begin{cases} 1 & \text{if } S_j \neq S_k,\ j > k, \\ 0 & \text{otherwise,} \end{cases}$$

so that $Z_n = \Sigma_{k=1}^n V_k$ is the number of states visited by the random walk in time n which are never revisited. Note that R_n is the number of states visited in time n which are not revisited prior to time $n+1$ so that $R_n \geqslant Z_n$.

Now it has been shown by Spitzer et al. [9, p. 39] that

$$\lim_{n\to\infty} n^{-1} R_n = 1 - F \quad \text{a.s.}$$

where F is the probability of ultimate return to the origin for the random walk. In the course of proving this result, they note that V_0,

$V_1, \ldots$ is a stationary ergodic sequence so that it follows from the ergodic theorem that

$$\lim_{n\to\infty} n^{-1} Z_n = \mathrm{E} V_0 = \mathrm{P}\{S_1 \neq 0, S_2 \neq 0, \ldots\} = 1 - F \text{ a.s.}$$

Of course $Z_n = 0$ a.s. for recurrent random walk ($F = 1$).

In this work, we shall confine consideration to the case of transient random walk. We shall obtain central limit and iterated logarithm results for the processes $\{R_n\}$ and $\{Z_n\}$.

Various central limit results for R_n for random walks taking values in the d-dimensional space of integer lattice points have previously been obtained by Jain and Orey [3] (general d), Jain and Pruitt [4] ($d = 3, 4$). One of our results (Corollary 1) overlaps with Theorem 1 of [3], but our methods are different. We have restricted consideration to the case $d = 1$ in the interests of a unified exposition; some of our results continue to hold for general d.

2. Central limit results for Z_n

Let $f_n = \mathrm{P}\{S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0\}, n > 1; f_1 = \mathrm{P}\{S_1 = 0\}$. Here f_n denotes the probability that the first return to zero should occur at time n. We shall call the random walk *strongly transient* (see [3]) if $\Sigma_{j=1}^{\infty} j f_j < \infty$ or equivalently $\Sigma_{j=0}^{\infty} \Sigma_{n=j+1}^{\infty} f_n < \infty$. We start by obtaining the following result:

Theorem 2.1. *For a strongly transient random walk,* $\sigma^2 = \lim_{n\to\infty} n^{-1} \operatorname{var} Z_n$ *exists and* $F(1-F) \leqslant \sigma^2 < \infty$. *Also, as* $n \to \infty$,

$$\frac{Z_n - n(1-F)}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, 1).$$

("$\mathcal{D}$" denotes convergence in distribution.)

Proof. Let T be the ergodic measure preserving transformation on our basic probability space such that $X_k(\omega) = X(\mathrm{T}^k \omega)$. It is known that $\{V_0, V_1, \ldots\}$ is stationary and ergodic and in fact $V_k(\omega) = V_0(\mathrm{T}^k \omega)$ as is easily seen by defining

$$\phi(X_1, X_2, \ldots) = \begin{cases} 1 \text{ if } S_k \neq 0, k \geqslant 1, \\ 0 \text{ otherwise,} \end{cases}$$

and noting that

$$V_k = \phi(X_k, X_{k+1}, \ldots).$$

Write M_m for the σ-field generated by $X_m, X_{m+1}, \ldots$. Then $M_m = T^{-m}(M_0)$. We shall obtain the result of the theorem by applying Theorem 2 of [1] (case $\delta = \infty$).

First we need to check that

$$(1) \qquad \sum_{m=1}^{\infty} \mathbf{E}\,|\,\mathbf{E}\{V_0 - \mathbf{E}V_0 \,|\, M_m\}\,| < \infty.$$

We have for $m > 1$,

$$
\begin{aligned}
(2) \qquad \mathbf{E}(V_0 \,|\, M_m) &= \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ \ldots \,|\, X_m, X_{m+1}, \ldots\} \\
&\leqslant \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots, S_{m-1} \neq 0 \,|\, X_m, X_{m+1}, \ldots\} \\
&= \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots, S_{m-1} \neq 0\},
\end{aligned}
$$

and $\mathbf{E}\{\mathbf{E}(V_0 \,|\, M_m)\} = \mathbf{E}V_0$, so that using (2),

$$
\begin{aligned}
&\mathbf{E}|\mathbf{E}(V_0 \,|\, M_m) - \mathbf{E}V_0| = \\
&= 2 \int_{\{\mathbf{E}(V_0 | M_m) > \mathbf{E}V_0\}} (\mathbf{E}(V_0 \,|\, M_m) - \mathbf{E}V_0)\mathrm{d}\mathbf{P} \\
&\leqslant 2(\mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots, S_{m-1} \neq 0\} - \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots\}) \\
&= 2\left(\left(1 - \sum_{j=1}^{m-1} f_j\right) - \left(1 - \sum_{j=1}^{\infty} f_j\right)\right) = 2\sum_{j=m}^{\infty} f_j,
\end{aligned}
$$

and hence (1) holds in view of the strong transience condition. Theorem 2 of [1] then gives that

$$\lim_{n \to \infty} (n^{-1} \operatorname{var} Z_n) = \sigma^2,$$

with $0 \leqslant \sigma < \infty$ and if $\sigma > 0$, the required normal convergence result holds.

To obtain a usefully concrete general expression for σ does not seem to be possible, but it is not difficult to show that $\sigma^2 \geqslant F(1-F)$. We have

$$(3) \qquad \sigma^2 = \mathrm{E}(V_0 - \mathrm{E}V_0)^2 + 2\sum_{k=1}^{\infty} \mathrm{E}(V_0 - \mathrm{E}V_0)(V_k - \mathrm{E}V_k)$$

$$= F(1-F) + 2\sum_{k=1}^{\infty} \{\mathrm{E}(V_0 V_k) - (1-F)^2\}$$

$$= F(1-F) + 2\sum_{k=1}^{\infty} (\mathbf{P}\{V_0 = 1, V_k = 1\} - (1-F)^2)$$

and

$$(4) \qquad \mathbf{P}\{V_0 = 1, V_k = 1\} = \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots; \; S_{k+1} \neq S_k, S_{k+2} \neq S_k, \ldots\}$$

$$= \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots; \quad X_{k+1} \neq 0, X_{k+1} + X_{k+2} \neq 0, \ldots\}$$

$$= \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \mathbf{P}\{S_1 \neq 0, \ldots, S_{k-1} \neq 0, S_k = j; \; X_{k+1} \notin \{0, -j\}, X_{k+1} + X_{k+2} \notin \{0, -j\}, \ldots\}$$

$$= \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \mathbf{P}\{S_1 \neq 0, \ldots, S_{k-1} \neq 0, S_k = j\} \times \mathbf{P}\{S_1 \notin \{0, -j\}, S_2 \notin \{0, -j\}, \ldots\}.$$

Define, following Spitzer [9, Chapter VI],

$$\Pi_{\{0,-j\}}(0, y) = \mathbf{P}\{S_T = y; T < \infty\}, \quad y = 0, -j,$$

where

$$T = \min\{k \mid 1 \leqslant k \leqslant \infty, \; S_k \in \{0, -j\}\},$$

and

$$(5) \qquad E_{\{0,-j\}}(0) = 1 - \Pi_{\{0,-j\}}(0,0) - \Pi_{\{0,-j\}}(0,-j)$$

$$= \mathbf{P}\{S_1 \notin \{0,-j\}, S_2 \notin \{0,-j\}, \ldots\}.$$

Define also

$$G(x, y) = \sum_{n=0}^{\infty} \mathbf{P}\{S_n = y - x\}$$

($G(x, y) \leqslant G(0, 0) < \infty$ since the random walk is transient; see [9, p. 6,7].) Then, using [9, P2(b), p. 292], we have

$$\begin{aligned} 1 &= (1 - \Pi_{\{0,-j\}}(0, 0))\, G(0, 0) - \Pi_{\{0,-j\}}(0,-j)\, G(-j, 0), \\ 0 &= (1 - \Pi_{\{0,-j\}}(0, 0))\, G(0,-j) - \Pi_{\{0,-j\}}(0,-j)\, G(0, 0), \end{aligned} \tag{6}$$

and from (5) and (6) we readily find that

$$E_{\{0,-j\}}(0) = \frac{G(0,0) - G(0,-j)}{G^2(0,0) - G(0,-j)\, G(-j,0)}\,.$$

But, for transient random walk,

$$\frac{G(x, y)}{G(0, 0)} = F(x, y), \quad x \neq y,$$

where $F(x, y)$ represents the probability of the random walk ever reaching y starting from x [9, p. 10] and

$$G(0, 0) = (1 - F)^{-1}$$

[9, p. 7] so that

$$\begin{aligned} E_{\{0,-j\}}(0) &= (1 - F)\, \frac{1 - F(0,-j)}{1 - F(0,-j) F(-j, 0)} \\ &\geqslant (1 - F)\,(1 - F(0,-j)). \end{aligned} \tag{7}$$

Now let $f_n(x, y)$ denote the probability that the random walk makes a first passage from x to y at time n. Then, from (4), (5) and (7),

$$\begin{aligned} \mathbf{P}\{V_0 = 1, V_k = 1\} &\geqslant (1 - F) \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \mathbf{P}\{S_1 \neq 0, \ldots, S_{k-1} \neq 0, S_k = j\} \\ &\quad \times (1 - F(0,-j)) = (1 - F)\Big(\mathbf{P}\{S_1 \neq 0, \ldots, S_k \neq 0\} \\ &\quad - \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \mathbf{P}\{S_1 \neq 0, \ldots, S_{k-1} \neq 0, S_k = j\}\, F(j, 0)\Big), \end{aligned} \tag{8}$$

(8) $$\mathbf{P}\{V_0 = 1, V_k = 1\} \geqslant (1-F)\Big(1 - \sum_{j=1}^{k} f_j - \sum_{l=1}^{\infty} \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \mathbf{P}\{S_1 \neq 0, \ldots, S_{k-1} \neq 0, S_k = j\} f_l(j, 0)\Big)$$

$$= (1-F)\Big(1 - \sum_{j=1}^{k} f_j - \sum_{l=1}^{\infty} f_{k+l}\Big) = (1-F)^2$$

and hence, from (3) and (8), $\sigma^2 \geqslant F(1-F)$. This completes the proof of the theorem.

3. Functional central limit and iterated logarithm results for Z_n

If we impose certain more stringent conditions than strong transience on the random walk, we can obtain a functional form of the central limit theorem for appropriate random functions in the space $D[0, 1]$ and also an iterated logarithm law. Here $D[0, 1]$ is the space of functions on the interval $[0, 1]$ with at most discontinuities of the first kind.

Let

$$\rho(x, y) = \sup_{0 \leqslant t \leqslant 1} |x(t) - y(t)| \quad \text{for } x, y \in D,$$

and define a sequence of real random functions $\xi_n(\cdot)$ belonging to D by

$$\xi_n(t) = [Z_n - Z_{n-j-1} - (j+1)(1-F)]/(\operatorname{var} Z_n)^{1/2},$$

$$j \leqslant nt < j+1, \; j = 0, 1, \ldots, n-1,$$

$$\xi_n(1) = [Z_n - n(1-F)]/(\operatorname{var} Z_n)^{1/2}.$$

Theorem 3.1. *If* $\Sigma_{n=1}^{\infty} (\Sigma_{j=n}^{\infty} f_j)^{1/2} < \infty$, *then*

$$\xi_n \xrightarrow{\mathcal{D}} W$$

in the sense (D, ρ), *where* W *is a standard Wiener process.*

Proof. The result of this theorem is a straightforward application of Theorem 4 of [7]. All that needs to be checked is

$$(9) \qquad \sum_{m=1}^{\infty} [\mathrm{E}\{\mathrm{E}(V_0 - \mathrm{E}V_0 \mid M_m)\}^2]^{1/2} < \infty;$$

that $\sigma^2 = \lim_{n\to\infty} \{n^{-1} \operatorname{var} Z_n\}$ exists with $\sigma > 0$ follows from Theorem 2.1. Now, from the proof of Theorem 2.1 we see that

$$\mathrm{E}(V_0 \mid M_m) \leqslant 1 - \sum_{j=1}^{m-1} f_j$$

for $m > 1$, so that then

$$0 \leqslant \mathrm{E}(\mathrm{E}(V_0 - \mathrm{E}V_0 \mid M_m))^2 = \mathrm{E}(\mathrm{E}(V_0 \mid M_m))^2 - (\mathrm{E}V_0)^2$$

$$\leqslant \left(1 - \sum_{j=1}^{m-1} f_j\right)^2 - \left(1 - \sum_{j=1}^{\infty} f_j\right)^2$$

$$\leqslant 2\left(\left(1 - \sum_{j=1}^{m-1} f_j\right) - \left(1 - \sum_{j=1}^{\infty} f_j\right)\right) = 2 \sum_{j=m}^{\infty} f_j$$

and (9) follows under the specified condition $\Sigma_{n=1}^{\infty} (\Sigma_{j=n}^{\infty} f_j)^{1/2} < \infty$.

Next we have the following iterated logarithm result.

Theorem 3.2. *If $\Sigma_{j=n}^{\infty} f_j = O(n^{-2-\delta})$ as $n \to \infty$ for some $\delta > 0$ then $\sigma^2 = \lim_{n\to\infty} \{n^{-1} Z_n\}$ exists with $F(1-F) \leqslant \sigma^2 < \infty$ and*

$$n^{-1} Z_n = 1 - F + \xi(n)(2\sigma^2 n^{-1} \log\log n)^{1/2}$$

where $\xi(n)$ has its set of limit points confined to $[-1, 1]$ and $\limsup_{n\to\infty} \xi(n) = 1$ a.s., $\liminf_{n\to\infty} \xi(n) = -1$ a.s.

Proof. The result follows from an application of [5, Theorem 6]. $\{X_n\}$ is stationary and satisfies the uniformly strong mixing condition with mixing coefficients of zero. Also, $V_n \leqslant 1$ for all n, so in order to apply Theorem 6 of [5] it just remains to show that

$$\text{(10)} \qquad \mathrm{E}(V_0 - \mathrm{E}(V_0 \mid \mathcal{M}_1^k))^2 = \mathrm{O}(k^{-2-\delta}),$$

for some $\delta > 0$, where $\mathcal{M}_1^k$ is the σ-field generated by $X_1, ..., X_k$. Now

$$\text{(11)} \qquad \mathrm{E}(V_0 - \mathrm{E}(V_0 \mid \mathcal{M}_1^k))^2 = \mathrm{E}V_0^2 - \mathrm{E}(\mathrm{E}(V_0 \mid \mathcal{M}_1^k))^2$$
$$= 1 - F - \mathrm{E}(\mathrm{E}(V_0 \mid \mathcal{M}_1^k))^2$$

and

$$\mathrm{E}(V_0 \mid \mathcal{M}_1^k) = \mathrm{P}\{S_1 \neq 0, S_2 \neq 0, .. \mid X_1, ..., X_k\}$$
$$= \begin{cases} 0 & \text{if } 0 \in \{S_1, ..., S_k\}, \\ 1 - F(0, -S_k) & \text{if } 0 \notin \{S_1, ..., S_k\}, \end{cases}$$

so that

$$\text{(12)} \qquad \mathrm{E}(\mathrm{E}(V_0 \mid \mathcal{M}_1^k))^2 = \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} (1 - F(0,-j))^2 \, \mathrm{P}\{S_1 \neq 0, ..., S_{k-1} \neq 0, S_k = j\}$$
$$\geqslant \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} (1 - 2F(0,-j)) \mathrm{P}\{S_1 \neq 0, ..., S_{k-1} \neq 0, S_k = j\}$$
$$= 1 - F - \sum_{j=k+1}^{\infty} f_j,$$

using results obtained in the process of establishing (8). Thus, from (11) and (12),

$$\mathrm{E}(V_0 - \mathrm{E}(V_0 \mid \mathcal{M}_1^k))^2 \leqslant \sum_{j=k+1}^{\infty} f_j$$

and (10) follows. The result of the theorem is then immediate.

4. Limit results for R_n

The limit behaviour of R_n is closely related to that of Z_n and can be derived from it with the aid of the following theorem.

Theorem 4.1. *For strongly transient random walk*

$$n^{-1/2}(R_n - Z_n) \xrightarrow{p} 0$$

as $n \to \infty$, "p" denoting convergence in probability. If $\Sigma_{j=n}^{\infty} f_j = O(n^{-2-\delta})$ as $n \to \infty$ for some $\delta > 0$, then

$$n^{-1/2}(R_n - Z_n) \xrightarrow{\text{a.s.}} 0.$$

Proof. Write $R_n = \Sigma_{k=1}^{n} W_{k,n}$, where $W_{n,n} = 1$, and

$$W_{k,n} = \begin{cases} 1 & \text{if } S_j \neq S_k, \ k < j \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

$1 \leqslant k < n$; note that $W_{k,n} \geqslant V_k$, $k = 1, 2, \ldots, n$. We have for $k < n$

$$\begin{aligned} \mathrm{E}W_{k,n} &= \mathrm{P}\{S_{k+1} \neq S_k, S_{k+2} \neq S_k, \ldots, S_n \neq S_k\} \\ &= \mathrm{P}\{X_{k+1} \neq 0, X_{k+1} + X_{k+2} \neq 0, \ldots, X_{k+1} + X_{k+2} + \ldots + X_n \neq 0\} \\ &= \mathrm{P}\{S_1 \neq 0, S_2 \neq 0, \ldots, S_{n-k} \neq 0\} = 1 - \sum_{j=1}^{n-k} f_j, \end{aligned}$$

so that

$$\begin{aligned} n^{-1/2}\mathrm{E}|R_n - Z_n| &= n^{-1/2}\mathrm{E}(R_n - Z_n) \\ &= n^{-1/2}\sum_{k=1}^{n} (\mathrm{E}W_{k,n} - 1 + F) = n^{-1/2}\sum_{k=1}^{n}\sum_{j=n-k+1}^{\infty} f_j \\ &= n^{-1/2}\sum_{l=1}^{n}\sum_{j=l}^{\infty} f_j \to 0 \end{aligned}$$

as $n \to \infty$ and $n^{-1/2}(R_n - Z_n) \xrightarrow{p} 0$ follows from an application of Markov's inequality.

Now suppose that $\Sigma_{j=n}^{\infty} f_j = O(n^{-2-\delta})$ as $n \to \infty$ for some $\delta > 0$. Then, there is a constant $C > 0$ such that $\Sigma_{j=n}^{\infty} f_j \leqslant C n^{-2-\delta}$ for every n. Let $k(n)$ be integer valued with $k(n) \uparrow \infty$ and $n - k(n) \uparrow \infty$ as $n \to \infty$. Then

$$\text{(13)}\qquad \mathrm{E}\Big|\sum_{k=1}^{k(n)} (W_{k,n}-V_k)\Big| = \mathrm{E}\sum_{k=1}^{k(n)} (W_{k,n}-V_k) = \sum_{k=1}^{k(n)} \sum_{j=n-k+1}^{\infty} f_j$$

$$\leqslant C\sum_{k=1}^{k(n)} (n-k+1)^{-2-\delta} \leqslant C\int_1^{k(n)+1} (n-x+1)^{-2-\delta}\,\mathrm{d}x$$

$$\sim C(1+\delta)^{-1}(n-k(n))^{-1-\delta} \quad \text{as } n\to\infty .$$

Now choose $k(n) = n-[n^{\beta}]$, where $[x]$ denotes the integer part of x and β is chosen so that $\frac{1}{2} > \beta > \frac{1}{2}(1+\delta)^{-1}$. Then, from (13),

$$\sum_{n=1}^{\infty} n^{-1/2}\,\mathrm{E}\Big|\sum_{k=1}^{k(n)} (W_{k,n}-V_k)\Big| < \infty$$

and hence $n^{-1/2}\sum_{k=1}^{k(n)} (W_{k,n}-V_k) \xrightarrow{\text{a.s.}} 0$ using Markov's inequality and the Borel–Cantelli lemma. Further,

$$0 \leqslant n^{-1/2}\sum_{k=k(n)+1}^{n} (W_{k,n}-V_k) \leqslant n^{-1/2}[n^{\beta}] \to 0 \quad \text{as } n\to\infty$$

since $\beta < \frac{1}{2}$, so that

$$n^{-1/2}\sum_{k=k(n)+1}^{n} (W_{k,n}-V_k) \xrightarrow{\text{a.s.}} 0 .$$

Thus,

$$n^{-1/2}(R_n-Z_n) = n^{-1/2}\sum_{k=1}^{k(n)} (W_{k,n}-V_k)$$

$$+\, n^{-1/2}\sum_{k=k(n)+1}^{n} (W_{k,n}-V_k) \xrightarrow{\text{a.s.}} 0$$

as required. This completes the proof of the theorem.

Corollary 4.2. *For a strongly transient random walk,* $\sigma^2 = \lim_{n\to\infty} n^{-1}\operatorname{var} R_n = \lim_{n\to\infty} n^{-1}\operatorname{var} Z_n$ *exists (see Theorem 2.1) and* $F(1-F) \leqslant \sigma^2 < \infty$. *Also, as* $n\to\infty$,

$$\text{(14)}\qquad \frac{R_n - n(1-F)}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1).$$

if $\Sigma_{j=n}^{\infty} f_j = O(n^{-2-\delta})$ *as* $n \to \infty$ *for some* $\delta > 0$, *then*

(15) $$n^{-1} R_n = 1 - F + \eta(n)\,(2\sigma n^{-1} \log\log n)^{1/2}$$

where $\eta(n)$ *has its set of limit points confined to* $[-1, 1]$ *and* $\limsup_{n\to\infty} \eta(n) = 1$ a.s., $\liminf_{n\to\infty} \eta(n) = -1$ a.s.

Proof. We remark that the result (14) has been proved directly, except for the identification of $\lim_{n\to\infty} (n^{-1} \operatorname{var} R_n)$ and $\lim_{n\to\infty} (n^{-1} \operatorname{var} Z_n)$, by Jain and Orey [3]. Here (14) and (15) follow immediately from Theorems 2.1, 3.2, 4.1 with $\sigma^2 = \lim_{n\to\infty} (n^{-1} \operatorname{var} Z_n)$, and the only thing that remains to be checked is that $\sigma^2 = \lim_{n\to\infty} (n^{-1} \operatorname{var} R_n)$.

Now, from [3, Theorem 1], $\lim_{m\to\infty} (m^{-1} \operatorname{var} R_m)$ exists and is positive (and finite) for strongly transient random walk and furthermore,

$$(n \lim_{m\to\infty} (m^{-1} \operatorname{var} R_m))^{-1/2} (R_n - n(1-F)) \xrightarrow{\mathcal{D}} N(0, 1).$$

We can thus identify $\lim_{m\to\infty} (m^{-1} \operatorname{var} R_m)$ as σ^2. An expression for $\lim_{m\to\infty} (m^{-1} \operatorname{var} R_m)$ is given in [3] but it is not obvious that it coincides with (3).

Corollary 4.3. *Let*

$$\xi_n'(t) = (\operatorname{var} Z_n)^{-1/2} (R_n - R_{n-j-1} - (j+1)(1-F)),$$

$$j \leq nt < j+1,\ j = 0, 1, \ldots, n-1\ ,$$

$$\xi_n'(1) = (\operatorname{var} Z_n)^{-1/2} (R_n - n(1-F)).$$

If $\Sigma_{j=n}^{\infty} f_j = O(n^{-2-\delta})$ *as* $n \to \infty$ *for some* $\delta > 0$, *then*

$$\xi_n' \xrightarrow{\mathcal{D}} W$$

in the sense (D, ρ) *where* W *is a standard Wiener process.*

Proof. In order to deduce this result from Theorem 3.1 it suffices to show that $\rho(\xi_n, \xi_n') \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. This is simply obtained with the aid of Theorem 4.1. That $n^{-1/2} \max_{1 \leq k \leq n} (R_k - Z_k) \xrightarrow{\text{a.s.}} 0$ follows from $n^{-1/2}(R_n - Z_n) \xrightarrow{\text{a.s.}} 0$ is an immediate consequence of the following elementary result: Let $a_1, a_2, \ldots$ and $0 < b_1 \leqslant b_2 \leqslant \ldots$ be real numbers with $b_n \to \infty$ and $a_n/b_n \to 0$; then $\max_{1<k<n} a_k/b_n \to 0$.

Remark. It is quite possible that $\operatorname{var} Z_n$ may be replaced by, say, $\sigma^2 n$ in the norming of the random functions ξ_n, ξ'_n. To accomplish this would involve the obtaining of a more detailed asymptotic result for $\operatorname{var} Z_n$ than $\operatorname{var} Z_n \sim \sigma^2 n$. However, our present form seems equally useful for the obvious applications.

5. Some sufficient conditions involving moments

In the various theorems above, the conditions $\Sigma_{n=1}^{\infty} (\Sigma_{j=n}^{\infty} f_j) < \infty$, $\Sigma_{n=1}^{\infty} (\Sigma_{j=n}^{\infty} f_j)^{1/2} < \infty$ and $\Sigma_{j=n}^{\infty} f_j = O(n^{-2-\delta})$, some $\delta > 0$, figure. It is our object in this section to obtain moment conditions which ensure that these are satisfied. Here we shall suppose that $\mathrm{E}|X_1| < \infty$ so that, necessarily for transience of the random walk, $\mathrm{E}X_1 \neq 0$. We write $x^+ = \max(0,x)$, $x^- = -\min(0,x)$.

Theorem 5.1. *Suppose* $\mathrm{E}|X_1| < \infty$ *and* $\mathrm{E}X_1 \neq 0$. *Then* $\Sigma_{n=1}^{\infty} (\Sigma_{j=n}^{\infty} f_j) < \infty$ *if either* $\mathrm{E}X_1 > 0$, $\mathrm{E}(X_1^-)^2 < \infty$ *or* $\mathrm{E}X_1 < 0$, $\mathrm{E}(X_1^+)^2 < \infty$. *Also,* $\Sigma_{j=n}^{\infty} f_j = O(n^{-5/2})$ *and hence* $\Sigma_{n=1}^{\infty} (\Sigma_{j=n}^{\infty} f_j)^{1/2} < \infty$ *if* $\mathrm{E}X_1^4 < \infty$.

Proof. We take $\mathrm{E}X_1 > 0$, as we can do without loss of generality (replacing the X_i by $-X_i$ if necessary). Using standard recurrent event theory (noting that returns to the origin of the random walk constitutes a recurrent event process) we define

$$U(t) = \sum_{n=0}^{\infty} \mathrm{P}\{S_n = 0\}\, t^n, \qquad 0 \leqslant t < 1,$$

$$F(t) = \sum_{n=1}^{\infty} f_n t^n, \qquad 0 \leqslant t < 1,$$

and recall the identity

$$U(t)\,(1-F(t)) = 1.$$

From this identity it is clear that $[\mathrm{d}F(t)/\mathrm{d}t]_{t=1-} < \infty$ if and only if $[\mathrm{d}U(t)/\mathrm{d}t]_{t=1-} < \infty$. That is, $\Sigma_{j=1}^{\infty} j f_j < \infty$ (strong transience) if and only if $\Sigma_{j=1}^{\infty} j\,\mathbf{P}\{S_j = 0\} < \infty$. Furthermore, it has been shown by Spitzer [8] that

$$(16) \qquad 1 + \sum_1^{\infty} \mathbf{P}\{M_{n-1} \leqslant 0, S_n = 0\}\, t^n$$

$$= \exp\left[\sum_1^{\infty} t^n n^{-1} \mathbf{P}\{S_n = 0\}\right], \quad 0 \leqslant t < 1,$$

where $M_0 = 0$, $M_n = \max_{1 \leqslant k \leqslant n} S_k$, $n \geqslant 1$, and the events $\{M_{n-1} \leqslant 0, S_n = 0\}$ signify returns to the origin prior to the first entry into $[1, \infty)$ and thus define a recurrent event situation. Consequently,

$$1 + \sum_{n=1}^{\infty} \mathbf{P}\{M_{n-1} \leqslant 0, S_n = 0\}\, t^n$$

$$= \left(1 - \mathbf{P}\{S_1 = 0\}\, t - \sum_{n=2}^{\infty} \mathbf{P}\{M_{n-1} < 0, S_n = 0\}\, t^n\right)^{-1},$$

$$0 \leqslant t < 1,$$

and from (16),

$$\mathbf{P}\{S_1 = 0\}\, t + \sum_{n=2}^{\infty} \mathbf{P}\{M_{n-1} < 0, S_n = 0\}\, t^n$$

$$= 1 - \exp\left[-\sum_{n=1}^{\infty} t^n n^{-1} \mathbf{P}\{S_n = 0\}\right], \quad 0 \leqslant t < 1.$$

Thus, upon differentiating twice and letting $t \uparrow 1$ we find that $\Sigma_{j=1}^{\infty} j\mathbf{P}\{S_j = 0\} < \infty$ if and only if $\Sigma_{n=2}^{\infty} n^2\, \mathbf{P}\{M_{n-1} < 0, S_n = 0\} < \infty$. But $\Sigma_{n=2}^{\infty} n^2\, \mathbf{P}\{M_{n-1} < 0, S_n = 0\} < \infty$ if $\Sigma_{n=2}^{\infty} n^2\, \mathbf{P}\{M_{n-1} < 0, S_n \geqslant 0\} < \infty$ and

$$\mathbf{P}\{M_{n-1} < 0, S_n \geqslant 0\} = \mathbf{P}\{M_{n-1} < 0\} - \mathbf{P}\{M_n < 0\}$$

from which we find that $\Sigma_{n=2}^{\infty} n^2\, \mathbf{P}\{M_{n-1} < 0, S_n \geqslant 0\} < \infty$ if $\Sigma_{n=1}^{\infty} n\mathbf{P}\{M_n < 0\} < \infty$. Now it is a well-known result due to Sparre-Andersen that (e.g. [9, p. 219])

$$1 + \sum_{n=1}^{\infty} \mathbf{P}\{M_n < 0\}\, t^n = \exp\left[\sum_{n=1}^{\infty} t^n n^{-1} \mathbf{P}\{S_n < 0\}\right], \quad 0 \leqslant t < 1,$$

and we deduce from this that $\Sigma_{n=1}^{\infty} n\mathbf{P}\{M_n < 0\} < \infty$ if and only if $\Sigma_{n=1}^{\infty} \mathbf{P}\{S_n < 0\} < \infty$. Finally, Heyde [2] has shown that (under $\mathbf{E}|X_1| < \infty$, $\mathbf{E}X_1 > 0$) $\Sigma_{n=1}^{\infty} \mathbf{P}\{S_n < 0\} < \infty$ if and only if $\mathbf{E}(X^-)^2 < \infty$. This provides the required sufficient condition for strong transience.

To deal with the other conditions, note first that $f_j \leqslant \mathbf{P}\{S_j = 0\}, j \geqslant 1$. Furthermore, using a local limit theorem of Petrov [6], we readily find, under the condition $\mathbf{E}X_1^4 < \infty$, that there exists a constant $C > 0$ such that

$$\mathbf{P}\{S_j = 0\} \leqslant Cj^{7/2}$$

for $j \geqslant 1$. Then,

$$\sum_{j=n}^{\infty} \mathbf{P}\{S_j = 0\} \leqslant C \sum_{j=n}^{\infty} j^{-7/2} \sim 2C/5n^{5/2}$$

which yields the required results.

6. Some examples

It is difficult to obtain $\sigma^2 = \lim_{n\to\infty} (n^{-1} \operatorname{var} Z_n)$ in a satisfying explicit form without some simplifying assumption such as that the random walk is left continuous with negative mean or right continuous with positive mean [9, p. 289]. In other cases it does not seem possible to obtain a tractable expression for $E_{\{0,-j\}}(0)$.

For illustrative purposes we consider the simplest case, that of a Bernoulli random walk,

$$\mathbf{P}\{X_i = +1\} = p, \quad \mathbf{P}\{X_i = -1\} = q = 1 - p,$$

and we shall suppose for convenience that $p > q$. In this case it is known [9, p. 12] that

$$F = F(0,0) = 2q,$$
$$F(0,x) = \begin{cases} 1, & x > 0, \\ (qp^{-1})^x, & x < 0, \end{cases}$$

so that

$$E_{\{0,-j\}}(0) = \begin{cases} p-q, & j > 0, \\ 0, & j < 0, \end{cases}$$

and hence

$$\mathbf{P}\{V_0 = 1,\ V_k = 1\} = (p-q)\,\mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots, S_{k-1} \neq 0, S_k > 0\}$$

$$= (p-q)\,\mathbf{P}\{S_1 > 0, \ldots, S_k > 0\}$$

since $S_i < 0$ for some $i < k$, $S_k > 0$ is not possible without a passage through zero. Now, since the random walk is both left and right continuous,

$$\begin{aligned} 1-F = p- \quad &= \mathbf{P}\{S_1 \neq 0, S_2 \neq 0, \ldots\} \\ &= \mathbf{P}\{S_1 > 0, S_2 > 0, \ldots\} + \mathbf{P}\{S_1 < 0, S_2 < 0, \ldots\} \\ &= \mathbf{P}\{S_1 > 0, S_2 > 0, \ldots\} \end{aligned}$$

because $S_n \to \infty$ with probability one as $n \to \infty$ (since $p > q$). Thus,

$$\sigma^2 = 2q(p-q) + 2(p-q) \sum_{k=1}^{\infty} (\mathbf{P}\{S_1 > 0, \ldots, S_k > 0\}$$

$$-\mathbf{P}\{S_1 > 0, S_2 > 0, \ldots\}).$$

Further, using another result of Sparre–Andersen (e.g. [9, p. 219]),

$$\text{(17)} \qquad 1 + \sum_{k=1}^{\infty} \mathbf{P}\{S_1 > 0, \ldots, S_k > 0\}\, t^k$$

$$= \exp\left[\sum_{k=1}^{\infty} t^k k^{-1}\, \mathbf{P}\{S_k > 0\}\right], \quad 0 \leqslant t < 1,$$

and, since the random walk is left continuous, we have [9, p. 186, 187]

$$\exp\left\{\sum_{k=1}^{\infty} t^k k^{-1}\, \mathbf{P}\{S_k > 0\}\right\} = \frac{tq(1-r(t))}{(1-t)r(t)}$$

where

$$r(t) = (2pt)^{-1}(1-\sqrt{1-4pqt^2})$$

and $r(1) = qp^{-1}$. Thus,

$$\sum_{k=1}^{\infty} (\mathbf{P}\{S_1 > 0, \dots, S_k > 0\} - \mathbf{P}\{S_1 > 0, S_2 > 0, \dots\})$$

$$= \lim_{t\uparrow 1} \left(\frac{tq(1-r(t))}{(1-t)r(t)} - (p-q)\frac{t}{1-t}\right) - 1$$

$$= \frac{p}{q}\lim_{t\uparrow 1}\left(\frac{q-pr(t)}{1-t}\right) - 1 = p^2q^{-1}r'(1) - 1 = q(p-q)^{-1}$$

and

$$\sigma^2 = 4pq.$$

Finally, we shall establish the following proposition.

Theorem 6.1. *In the case of a left (right) continuous random walk with* $\mathrm{E}|X_1| < \infty$, $\mathrm{E}X_1 < 0$ $(\mathrm{E}X_1 > 0)$, *the following three conditions are equivalent:*

(i) *the random walk is strongly transient*;

(ii) $\lim_{n\to\infty} n^{-1} \operatorname{var} Z_n = \sigma^2 < \infty$;

(iii) $\mathrm{E}(X_1^+)^2 < \infty$ $(\mathrm{E}(X_1^-)^2 < \infty)$.

Proof. First note that from Theorem 5.1, (iii) implies (i) and from Theorem 2.1, (i) implies (ii). It thus remains only to show that (ii) implies (iii) under the conditions of the theorem. This we do by showing that (ii) does not hold if (iii) does not hold.

Now note that, by stationarity,

$$n^{-1} \operatorname{var} Z_n = \mathrm{E}(V_0 - \mathrm{E}V_0)^2 + 2\sum_{k=1}^{n-1} (1-kn^{-1})\mathrm{E}(V_0-\mathrm{E}V_0)(V_k-\mathrm{E}V_k)$$

$$= F(1-F) + 2\sum_{k=1}^{n-1} (1-kn^{-1})(\mathbf{P}\{V_0 = 1, V_k = 1\} - (1-F)^2)$$

which is non-decreasing in n, it having been noted in the proof of Theorem 2.1 that $\mathbf{P}\{V_0 = 1, V_k = 1\} \geqslant (1-F)^2$. Furthermore, it is easily seen that $n^{-1} \operatorname{var} Z_n$ is unbounded above as n increases if $\sum_{n=1}^{\infty} (\mathbf{P}\{V_0 = 1, V_k = 1\} - (1-F)^2) = \infty$.

We consider the case of a left continuous random walk with negative mean; the results for a right continuous random walk with positive mean will follow by replacing the X_i's by $-X_i$'s. For such random walks, $F(0,j) = 1, j < 0$ [9, p. 289] and hence from (7),

$$\mathbf{E}_{\{0,-j\}}(0) = 1-F, \quad j<0.$$

Then,

$$\begin{aligned}
\mathbf{P}\{V_0=1,\ V_k=1\} &= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \mathbf{P}\{S_1\neq 0,\dots,S_{k-1}\neq 0, S_k=j\}\,\mathbf{E}_{\{0,-j\}}(0) \\
&\geqslant (1-F)\sum_{j=-\infty}^{-1} \mathbf{P}\{S_1\neq 0,\dots,S_{k-1}\neq 0,\ S_k=j\} \\
&= (1-F)\,\mathbf{P}\{S_1<0,\dots,S_k<0\},
\end{aligned}$$

upon making use of the property of left continuity. Also

$$\begin{aligned}
1-F &= \lim_{n\to\infty} \mathbf{P}\{S_1\neq 0,\ \dots,\ S_n\neq 0\} \\
&= \lim_{n\to\infty} \sum_{j=-\infty}^{-1} \mathbf{P}\{S_1\neq 0,\ \dots,\ S_{n-1}\neq 0,\ S_n=j\} \\
&\quad + \lim_{n\to\infty} \sum_{j=1}^{\infty} \mathbf{P}\{S_1\neq 0,\ \dots,\ S_{n-1}\neq 0,\ S_n=j\} \\
&= \lim_{n\to\infty} \mathbf{P}\{S_1<0,\ \dots,\ S_n<0\} = \mathbf{P}\{S_1<0, S_2<0\ \dots\}
\end{aligned}$$

since

$$\sum_{j=1}^{\infty} \mathbf{P}\{S_1\neq 0,\ \dots,\ S_{n-1}\neq 0,\ S_n=j\} \leqslant \mathbf{P}\{S_n>0\} \to 0$$

as $n\to\infty$ for $\mathbf{E}X_1<0$. We have

$$\text{(18)}\qquad \begin{aligned}
&\sum_{k=1}^{\infty} (\mathbf{P}\{V_0=1,\ V_k=1\} - (1-F)^2) \\
&\qquad\geqslant (1-F)\sum_{k=1}^{\infty} (\mathbf{P}\{S_1<0,\dots,S_k<0\} - (1-F)),
\end{aligned}$$

and (17) gives, upon replacing X_i's by $-X_i$'s,

$$1+\sum_{k=1}^{\infty} \mathbf{P}\{S_1<0, \ldots, S_k<0\}\, t^k=\exp\left[\sum_{k=1}^{\infty} t^k k^{-1} \mathbf{P}\{S_k<0\}\right]$$

so that

$$\begin{aligned} 1-F &= \mathbf{P}\{S_1<0, S_2<0, \ldots\} \\ &= \lim_{t \uparrow 1}(1-t) \sum_{k=1}^{\infty} \mathbf{P}\{S_1<0, \ldots, S_k<0\}\, t^k \\ &= \lim_{t \uparrow 1} \exp\left[-\sum_{k=1}^{\infty} t^k k^{-1} \mathbf{P}\{S_k \geqslant 0\}\right] \\ &= \exp\left[-\sum_{k=1}^{\infty} k^{-1} \mathbf{P}\{S_k \geqslant 0\}\right] \end{aligned}$$

Consequently,

$$\begin{aligned} 1+&\sum_{k=1}^{\infty}(\mathbf{P}\{S_1<0, \ldots, S_k<0\}-(1-F))\, t^k \\ &= \exp\left[\sum_1^{\infty} t^k k^{-1} \mathbf{P}\{S_k<0\}\right] \\ &\quad -t(1-t)^{-1} \exp\left[-\sum_1^{\infty} k^{-1} \mathbf{P}\{S_k \geqslant 0\}\right] \\ &= (1-t)^{-1} \exp\left[-\sum_1^{\infty} k^{-1} \mathbf{P}\{S_k \geqslant 0\}\right] \\ &\quad \times\left(\exp\left[\sum_1^{\infty} k^{-1}(1-t^k) \mathbf{P}\{S_k \geqslant 0\}\right]-t\right) \\ &\geqslant (1-t)^{-1} \exp\left[-\sum_1^{\infty} k^{-1} \mathbf{P}\{S_k \geqslant 0\}\right] \sum_1^{\infty} k^{-1}(1-t^k) \mathbf{P}\{S_k \geqslant 0\}, \end{aligned}$$

so that

$$\sum_{k=1}^{\infty}(\mathbf{P}\{S_1<0, \ldots, S_k<0\}-(1-F))=\infty$$

if $\Sigma_{k=1}^{\infty} \mathbf{P}\{S_k \geqslant 0\} = \infty$. This condition holds if $\mathrm{E}(X^+)^2 = \infty$ by [2, Theorem A] and the required result follows from (18).

Theorem 6.1 shows that the strong transience condition is the best possible in a certain sense for the existence of general central limit results of the simple type of Theorem 2.1 and Corollary 4.2.

Note added in proof

Results similar to those of Corrollaries 4.2 and 4.3 have been given by Jain and Pruitt in [4a, 4b]. Their approach is direct in contrast to that of the present paper.

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